

The quaternionic weighted zeta function of a graph

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Abstract. We establish the quaternionic weighted zeta function of a graph and its Study determinant expressions. For a graph with quaternionic weights on arcs, we define a zeta function by using an infinite product which is regarded as the Euler product. This is a quaternionic extension of the square of the Ihara zeta function. We show that the new zeta function can be expressed as the exponential of a generating function and that it has two Study determinant expressions, which are crucial for the theory of zeta functions of graphs.

1 Introduction

Zeta functions of graphs started from Ihara zeta functions of regular graphs by Ihara [8]. Originally, Ihara [8] presented p -adic Selberg zeta functions of discrete groups, and showed that its reciprocal is an explicit polynomial. Ihara also showed the logarithm of Ihara zeta function has an expression in the form of a generating function. Serre [13] pointed out that the Ihara zeta function is the zeta function of the quotient T/Γ (a finite regular graph) of the one-dimensional Bruhat-Tits building T (an infinite regular tree) associated with $GL(2, k_p)$. A zeta function of a regular graph G associated with a unitary representation of the fundamental group of G was developed by Sunada [16], [17]. Hashimoto [6] treated multivariable zeta functions of bipartite graphs. For a general graph, Hashimoto [6] gave a

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determinant expression for its Ihara zeta function by its edge matrix. Bass [2] generalized Ihara's result on the Ihara zeta function of a regular graph to an irregular graph, and showed that its reciprocal is again a polynomial. Various proofs of Bass' Theorem have been given by Stark and Terras [14], Foata and Zeilberger [5], Kotani and Sunada [9]. For the weighted type of zeta function of graphs, Hashimoto [7] introduced a zeta function of a graph with weights assigned to its edges. Stark and Terras [14] defined the edge zeta function of a graph with weights assigned to its arcs, and gave its determinant expression by using its edge matrix. Mizuno and Sato [11] introduced a special version of the edge zeta function of a graph, and defined the weighted zeta function of a graph by using arc weights and the variable t counting the length of cycles. Later, they called this zeta function the first weighted zeta function in order to distinguish it from another zeta function which Sato defined after. It is noteworthy that the expressions of those zeta functions by the exponentials of generating functions were important to deriving the determinant expressions in the studies above. Recently, Watanabe and Fukumizu [18] presented a determinant expression for the edge zeta function of a graph by using matrices with size of the number of its vertices. As described above, zeta functions of graphs have been investigated for half a century.

On the other hand, the quaternions were discovered by Hamilton in 1843. It can be considered as an extension of the complex numbers. However, quaternions do not commute mutually in general. For many years, a number of people, for example Cayley, Study, Moore, Dieudonné, Dyson, Mehta, Xie, Chen, have given different definitions of determinants of quaternionic matrices. Detailed accounts on the determinants of quaternionic matrices can be found in, for example, [1, 19]. In order to extend the zeta function of a graph to the case of quaternions, we will use the approach developed by Study [15]. The Study determinant is the unique, up to a real power factor, functional which satisfies the following three axioms [1]:

(A1) $d(\mathbf{A}) = 0 \Leftrightarrow \mathbf{A}$ is singular.

(A2) $d(\mathbf{AB}) = d(\mathbf{A})d(\mathbf{B})$ for all $\mathbf{A}, \mathbf{B} \in \text{Mat}(n, \mathbb{H})$.

(A3) If \mathbf{A}' is obtained from \mathbf{A} by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $d(\mathbf{A}') = d(\mathbf{A})$.

Therefore it is essential to investigate the relation between the Study determinant and the quaternionic zeta function of a graph. The advantage of this approach is that one can reduce a calculation of the Study determinant to that of the ordinary determinant. Its disadvantage is that the Study determinant is not an exact extension of the determinant but is rather that of the square of it.

Our aim in this paper is to define the quaternionic zeta function of a graph by using an infinite product, namely the Euler product, and to show its essential characteristics. Specifically, we derive its expression in the form of the exponential of a generating function and determine its two types of Study determinant expressions such as in [6] (Hashimoto type) and in [2] (Bass type). Both types are crucial for a zeta function of a graph. In order to obtain these results, we explain the formal power series on a monoid algebra for preparations. The formal series, which includes the formal power series, is treated in detail in [3]. Our approach in this material follows the manner in [5, 12]. Furthermore, we give an

account of some relations between their exponentials and logarithms which can be found in Chapter IV of [4]. Up to the present, no quaternionic extension of zeta functions of graphs has been appeared. Hence our results can be regarded as the first successful construction of a quaternionic zeta function of a graph.

The rest of the paper is organized as follows. Section 2 treats noncommutative formal series and monoids. The notion of Lyndon word is given, and some application of the factorization theorem is discussed. The conclusion (Proposition 2.3) plays an essential role in our investigation. In Section 3, we define the exponential and the logarithm of noncommutative formal power series and give some formulas of which we make use in Section 6. In Section 4, we explain the Study determinant of a quaternionic matrix and extend it to the matrix whose entries are formal power series with quaternionic coefficients. In Section 5, we provide a summary of the Ihara zeta function and the various zeta function of a graph, and present their determinant expressions. In Section 6, we define a quaternionic zeta function of a graph by using an infinite product, and give its expression in the form of the exponential of a generating function. In Section 7, we show that the quaternionic zeta functions has two types of Study determinant expressions. One (Theorem 7.1) is an analogy to Hashimoto type expression [6] and another (Theorem 7.2) is to Bass type [2].

2 Noncommutative formal power series and monoids

Let R be a commutative ring with unity, and A an algebra over R . $A[[t]]$ denotes the ring of formal power series in t with coefficients in A . Each element α of $A[[t]]$ is expressed as

$$\alpha = \sum_{k \geq 0} \alpha_k t^k \quad (\alpha_k \in A).$$

$A[[t]]$ can be equipped with the topology defined by the following manner. Let ω be the function defined as follows:

$$\begin{aligned} \omega : A[[t]] \times A[[t]] &\longrightarrow \mathbb{N} \cup \{\infty\} \\ \omega(\alpha, \beta) &= \inf\{n \in \mathbb{N} \mid \alpha_n \neq \beta_n\}. \end{aligned}$$

Then an ultrametric distance d_ω on $A[[t]]$ is given by $d_\omega(\alpha, \beta) = 2^{-\omega(\alpha, \beta)}$ and a topology on $A[[t]]$ is derived from d_ω .

Let G be a monoid. $R[G]$ denotes the *monoid algebra* of G over R . $R[G]$ is the set of formal sums $z = \sum_{g \in G} z_g g$, where $z_g \in R$ for each $g \in G$ and $z_g = 0$ for all but finitely many g . The addition in $R[G]$ is coefficient-wise, and the elements of R commute with the elements of G in the multiplication.

Let $X = \{x_1, \dots, x_N\}$ be a finite nonempty totally ordered set in which elements are arranged ascendingly. X^* denotes the free monoid generated by X . Let $<$ be the lexicographic order on X^* derived from the total order on X . For a word $w = x_{i_1} x_{i_2} \cdots x_{i_r} \in X^*$, r is called the *length* of w which is denoted by $|w|$. The length of the empty word is defined to be 0. A nonempty word w in X^* is called a *Lyndon word* if w is *prime*, namely, not a power w'^r of any other word w' for any $r \geq 2$, and is minimal in the cyclic rearrangements of w . We denote by L_X the set of Lyndon words in X^* . It is well known that any nonempty word w can be formed uniquely as a nonincreasing sequence of Lyndon words.

Theorem 2.1. *For any nonempty word $w \in X^*$, there exists a unique nonincreasing sequence of Lyndon words l_1, l_2, \dots, l_r such that $w = l_1 l_2 \cdots l_r$.*

Proof. For the proof, see for example [10]. \square

Let us consider $R[X^*][[t]]$. Since $(1 - lt)^{-1} = 1 + lt + (lt)^2 + \cdots$ for every $l \in X^*$ in $R[X^*][[t]]$, Theorem 2.1 implies:

$$\prod_{l \in L_X}^> (1 - lt^{|l|})^{-1} = \sum_{w \in X^*} wt^{|w|}, \quad (2.1)$$

in $R[X^*][[t]]$, where $\prod_{l \in L_X}^>$ means that the factors are multiplied in decreasing order. On the other hand, it follows that

$$\sum_{w \in X^*} wt^{|w|} = \{1 - (x_1 + \cdots + x_N)t\}^{-1}. \quad (2.2)$$

(2.1) and (2.2) imply the following equation:

$$\{1 - (x_1 + \cdots + x_N)t\}^{-1} = \prod_{l \in L_X}^> (1 - lt^{|l|})^{-1}. \quad (2.3)$$

From (2.3), we obtain:

Proposition 2.2.

$$1 - (x_1 + \cdots + x_N)t = \prod_{l \in L_X}^< (1 - lt^{|l|}), \quad (2.4)$$

where $\prod_{l \in L_X}^<$ means that the factors are multiplied in increasing order.

Proof. In order to show that

$$\left\{ \prod_{l \in L_X}^> (1 - lt^{|l|})^{-1} \right\} \left\{ \prod_{l \in L_X}^< (1 - lt^{|l|}) \right\} = 1, \quad (2.5)$$

we check that the coefficient of t^r of the left hand side is equal to 1 if $r = 0$ and 0 if $r > 0$. Since $\prod_{l \in L_X}^> (1 - lt^{|l|})^{-1} = \prod_{l \in L_X}^> (1 + lt^{|l|} + l^2 t^{2|l|} + \cdots)$, the coefficient of at most r -th power of t in the left hand side of (2.5) is the same as that of the product:

$$\left\{ \prod_{\substack{l \in L_X \\ |l| \leq r}}^> (1 + lt^{|l|} + l^2 t^{2|l|} + \cdots) \right\} \left\{ \prod_{\substack{l \in L_X \\ |l| \leq r}}^< (1 - lt^{|l|}) \right\},$$

for every nonnegative integer $r \geq 0$. This is a finite product since $|X| < \infty$ and therefore is equal to 1. Thus (2.5) holds. \square

Let $[n] = \{1, 2, \dots, n\}$ with the natural order and $[n] \times [n]$ the cartesian product of $[n]$ with the lexicographic order derived from the natural order on $[n]$. We set $X = \{x(i, j) \mid (i, j) \in [n] \times [n]\}$ with the total order derived from $[n] \times [n]$. For each matrix $\mathbf{A} = (a_{ij}) \in \text{Mat}(n, A)$, we define $\rho^{\mathbf{A}}$ to be the R -algebra homomorphism from $R[X^*]$ to $\text{Mat}(n, A)$ defined by $\rho^{\mathbf{A}}(x(i, j)) = a_{ij} \mathbf{E}_{ij}$, where \mathbf{E}_{ij} denotes the (i, j) -matrix unit. Let $\mathbf{A}(i, j) = a_{ij} \mathbf{E}_{ij}$. One can extend $\rho^{\mathbf{A}}$ to the R -algebra homomorphism $\rho_t^{\mathbf{A}}$ from $R[X^*][[t]]$ to $\text{Mat}(n, A)[[t]]$ by defining $\rho_t^{\mathbf{A}}(t) = t$. For each Lyndon word $l = (i_1, j_1)(i_2, j_2) \cdots (i_r, j_r) \in L_{[n] \times [n]}$, we put $\mathbf{A}_l = \mathbf{A}(i_1, j_1) \mathbf{A}(i_2, j_2) \cdots \mathbf{A}(i_r, j_r)$. Then we have

$$\mathbf{I}_n - \{\mathbf{A}(1, 1) + \mathbf{A}(1, 2) + \cdots + \mathbf{A}(n, n)\}t = \prod_{l \in L_{[n] \times [n]}}^{<} (\mathbf{I}_n - \mathbf{A}_l t^{|l|}). \quad (2.6)$$

However we easily see that

$$\mathbf{A}_l = \begin{cases} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{r-1} i_r} a_{i_r j_r} \mathbf{E}_{i_1 j_r} & \text{if } j_k = i_{k+1} \text{ for } k = 1, \dots, r-1, \\ \mathbf{O}_n \text{ (zero matrix of size } n) & \text{otherwise,} \end{cases}$$

and that $\mathbf{A}(1, 1) + \mathbf{A}(1, 2) + \cdots + \mathbf{A}(n, n) = \mathbf{A}$, hence we obtain:

Proposition 2.3. *Let $\mathbf{A} \in \text{Mat}(n, A)$. Then,*

$$\mathbf{I}_n - \mathbf{A}t = \prod_{\substack{(i_1, j_1) \cdots (i_r, j_r) \in L_{[n] \times [n]} \\ j_k = i_{k+1} \text{ (} k=1, \dots, r-1 \text{)}}^{<} (\mathbf{I}_n - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{r-1} i_r} a_{i_r j_r} \mathbf{E}_{i_1 j_r} t^r), \quad (2.7)$$

in $\text{Mat}(n, A)[[t]]$.

3 Exponentials and Logarithms of noncommutative formal power series

Let $K = \mathbb{Q}[[t]]$ and $X = \{x_1, \dots, x_N\}$ as in the previous section. Then $K[X^*] = \mathbb{Q}[X^*][[t]]$, the ring of formal power series with coefficients in $\mathbb{Q}[X^*]$ where t is considered as a central element in $K[X^*]$. Every $\alpha \in \mathbb{Q}[X^*][[t]]$ can be written as $\alpha = \sum_{k \geq 0} \alpha_k t^k$ with $\alpha_k \in \mathbb{Q}[X^*]$. For an α which has zero constant term, we define the *exponential* of α to be the element of $\mathbb{Q}[X^*][[t]]$ as follows:

$$\exp \alpha = \sum_{m \geq 0} \frac{1}{m!} \alpha^m. \quad (3.8)$$

Similarly, for an α whose constant term equals 1, we define the *logarithm* of α as follows:

$$\log \alpha = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (\alpha - 1)^n. \quad (3.9)$$

If $\alpha_0 = 1$, then $\alpha = 1 + \beta$ for some $\beta \in \mathbb{Q}[X^*][[t]]$ with zero constant term. Then we may write $\log \alpha = \log(1 + \beta) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \beta^n$ and $\log \alpha$ has zero constant term. Conversely, if β has zero constant term, then the constant term of $\exp \beta$ equals 1.

Theorem 3.1. Let $\alpha, \beta \in \mathbb{Q}[X^*][[t]]$.

(1) If $\alpha_0 = 1$, then $\exp \log \alpha = \alpha$.

(2) If β has zero constant term, then $\log \exp \beta = \beta$.

Proof. We shall prove (1). Letting $\alpha = 1 + \beta$, we have:

$$\exp \log \alpha = \exp \log(1 + \beta) = \sum_{m \geq 0} \frac{1}{m!} \left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \beta^n \right)^m. \quad (3.10)$$

We consider the partial sum of (3.10) as follows:

$$\sum_{m=0}^k \frac{1}{m!} \left(\sum_{n=1}^{\ell} \frac{(-1)^{n-1}}{n} \beta^n \right)^m. \quad (3.11)$$

If $k, \ell \geq r$, then the coefficient of at most r -th power of β in (3.10) is equal to that in (3.11). On the other hand, by the Taylor expansion we have:

$$1 + x = \exp \log(1 + x) = \sum_{m \geq 0} \frac{1}{m!} \left(\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} x^n \right)^m, \quad (3.12)$$

for a real number x with $|x| < 1$. Similar to (3.11), if $k, \ell \geq r$, then the coefficient of at most r -th power of x in (3.12) is equal to that in (3.13):

$$\sum_{m=0}^k \frac{1}{m!} \left(\sum_{n=1}^{\ell} \frac{(-1)^{n-1}}{n} x^n \right)^m. \quad (3.13)$$

Hence the coefficient of at most r -th power of β in (3.10) is equal to that of x in (3.12). Since r is an arbitrary nonnegative integer, the coefficient of any power of β in (3.10) coincide with the coefficient of the same power of x in (3.12). Moreover $\beta = \sum_{n \geq 1} \alpha_n t^n$ implies that the coefficient of at most r -th power of t is determined by the terms of at most r -th power of β in (3.10). Since r is arbitrary, we obtain that $\exp \log(1 + \beta) = 1 + \beta$ in $\mathbb{Q}[X^*][[t]]$. (2) can be proven in the same manner. \square

Proposition 3.2. Assume $\alpha, \beta \in \mathbb{Q}[X^*][[t]]$ have zero constant terms. If $\alpha\beta = \beta\alpha$, then $\exp(\alpha + \beta) = (\exp \alpha)(\exp \beta)$.

Proof. By easy calculations, we have:

$$\exp(\alpha + \beta) = \sum_{m \geq 0} \sum_{\substack{k, \ell \geq 0 \\ k + \ell = m}} \frac{1}{k! \ell!} \alpha^k \beta^\ell = (\exp \alpha)(\exp \beta).$$

\square

Proposition 3.3. Assume $\alpha, \beta \in \mathbb{Q}[X^*][[t]]$ satisfy $\alpha_0 = \beta_0 = 1$ and put $\alpha' = \alpha - 1$, $\beta' = \beta - 1$. If $\alpha'\beta' = \beta'\alpha'$, then $\log(\alpha\beta) = \log \alpha + \log \beta$.

Proof. Let $p = \log(1 + \alpha')$, $q = \log(1 + \beta')$. Both p and q have zero constant term. By Theorem 3.1 (1), $\exp p = 1 + \alpha'$, $\exp q = 1 + \beta'$. Since $pq = qp$, $(1 + \alpha')(1 + \beta') = (\exp p)(\exp q) = \exp(p + q)$ by Proposition 3.2. Thus $\log((1 + \alpha')(1 + \beta')) = \log \exp(p + q) = p + q = \log(1 + \alpha') + \log(1 + \beta')$ by Theorem 3.1 (2). \square

Corollary 3.4. *Assume $\alpha \in \mathbb{Q}[X^*][[t]]$ satisfies $\alpha_0 = 1$ and put $\alpha' = 1 - \alpha$. Then $\log \alpha^{-1} = -\log \alpha$ where $\alpha^{-1} = (1 - \alpha')^{-1} = 1 + \alpha' + (\alpha')^2 + \cdots$ is in $\mathbb{Q}[X^*][[t]]$.*

Proof. It follows from Proposition 3.3 that:

$$\begin{aligned} 0 &= \log 1 = \log((1 - \alpha')(1 - \alpha')^{-1}) \\ &= \log((1 - \alpha')(1 + \alpha' + (\alpha')^2 + \cdots)) \\ &= \log(1 - \alpha') + \log(1 + \alpha' + (\alpha')^2 + \cdots) \\ &= \log(1 - \alpha') + \log((1 - \alpha')^{-1}). \end{aligned}$$

\square

Let A be a \mathbb{Q} -algebra. We set $X = \{x(i, j) \mid (i, j) \in [n] \times [n]\}$ in the same manner as in the Section 2. For each matrix $\mathbf{A} = (a_{ij}) \in \text{Mat}(n, A)$, we define $\sigma^{\mathbf{A}}$ to be the \mathbb{Q} -algebra homomorphism from $\mathbb{Q}[X^*]$ to A defined by $\sigma^{\mathbf{A}}(x(i, j)) = a_{ij}$. It is clear that Theorem 3.1, Proposition 3.2, Proposition 3.3 and Corollary 3.4 hold even if one replaces $\mathbb{Q}[X^*]$ with A since one can extend $\sigma^{\mathbf{A}}$ to the \mathbb{Q} -algebra homomorphism $\sigma_t^{\mathbf{A}}$ from $\mathbb{Q}[X^*][[t]]$ to $A[[t]]$ by defining $\sigma_t^{\mathbf{A}}(t) = t$.

4 The Study determinant of a quaternionic matrix

Let \mathbb{H} be the set of quaternions. \mathbb{H} is a noncommutative associative algebra over \mathbb{R} , whose underlying real vector space has dimension 4 with a basis $1, i, j, k$ which satisfy the following relations:

$$i^2 = j^2 = k^2 = -1, \quad ij = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.$$

For $x = x_0 + x_1i + x_2j + x_3k \in \mathbb{H}$, x^* denotes the conjugate of x in \mathbb{H} which is defined by $x^* = x_0 - x_1i - x_2j - x_3k$, and $\text{Re } x = x_0$ the real part of x . One can easily check $xx^* = x^*x$, $(x^*)^n = (x^n)^*$, and $x^{-1} = x^*/|x|^2$ for $x \neq 0$. Hence, \mathbb{H} constitutes a skew field. We call $|x| = \sqrt{xx^*} = \sqrt{x^*x} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$ the norm of x . Indeed, $|\cdot|$ satisfies

- (1) $|x| \geq 0$, and moreover $|x| = 0 \Leftrightarrow x = 0$,
- (2) $|xy| = |x||y|$,
- (3) $|x + y| \leq |x| + |y|$.

Any quaternion x can be presented by two complex numbers $x = a + jb$ uniquely. Such a presentation is called the *symplectic decomposition*. Two complex numbers a and b are called the *simplex part* and the *perplex part* of x respectively. The symplectic decomposition is also valid for a quaternionic matrix, namely a matrix whose entries are quaternions. $\text{Mat}(m, n, \mathbb{H})$ denotes the set of $m \times n$ quaternionic matrices and $\text{Mat}(m, \mathbb{H})$ the set of $m \times m$

quaternionic matrices. For $\mathbf{M} \in \text{Mat}(m, n, \mathbb{H})$, we can write $\mathbf{M} = \mathbf{M}^S + j\mathbf{M}^P$ uniquely where $\mathbf{M}^S, \mathbf{M}^P \in \text{Mat}(m, n, \mathbb{C})$. \mathbf{M}^S and \mathbf{M}^P are called the *simplex part* and the *perplex part* of \mathbf{M} respectively. We define ψ to be the map from $\text{Mat}(m \times n, \mathbb{H})$ to $\text{Mat}(2m \times 2n, \mathbb{C})$ as follows:

$$\psi : \text{Mat}(m \times n, \mathbb{H}) \longrightarrow \text{Mat}(2m \times 2n, \mathbb{C}) \quad \mathbf{M} \mapsto \begin{bmatrix} \mathbf{M}^S & -\overline{\mathbf{M}^P} \\ \mathbf{M}^P & \overline{\mathbf{M}^S} \end{bmatrix},$$

where $\overline{\mathbf{A}}$ is the complex conjugate of a complex matrix \mathbf{A} . Then ψ is an \mathbb{R} -linear map. We also have:

Lemma 4.1. *Let $\mathbf{M} \in \text{Mat}(m \times n, \mathbb{H})$ and $\mathbf{N} \in \text{Mat}(n \times m, \mathbb{H})$. Then*

$$\psi(\mathbf{MN}) = \psi(\mathbf{M})\psi(\mathbf{N}).$$

Proof. Let $\mathbf{M} = \mathbf{A} + j\mathbf{B}$ and $\mathbf{N} = \mathbf{C} + j\mathbf{D}$ be the symplectic decompositions of \mathbf{M} and \mathbf{N} . Then,

$$\mathbf{MN} = (\mathbf{A} + j\mathbf{B})(\mathbf{C} + j\mathbf{D}) = \mathbf{AC} + \mathbf{A}j\mathbf{D} + j\mathbf{BC} + j\mathbf{B}j\mathbf{D}.$$

Since $\mathbf{X}j = j\overline{\mathbf{X}}$ for every complex matrix \mathbf{X} , we obtain:

$$\mathbf{MN} = \mathbf{AC} - \overline{\mathbf{B}}\mathbf{D} + j(\overline{\mathbf{A}}\mathbf{D} + \mathbf{BC}),$$

and therefore

$$\psi(\mathbf{MN}) = \begin{bmatrix} \mathbf{AC} - \overline{\mathbf{B}}\mathbf{D} & -\overline{\mathbf{A}}\mathbf{D} - \overline{\mathbf{BC}} \\ \overline{\mathbf{A}}\mathbf{D} + \mathbf{BC} & \overline{\mathbf{AC}} - \mathbf{B}\overline{\mathbf{D}} \end{bmatrix}.$$

On the other hand,

$$\psi(\mathbf{M})\psi(\mathbf{N}) = \begin{bmatrix} \mathbf{A} & -\overline{\mathbf{B}} \\ \mathbf{B} & \overline{\mathbf{A}} \end{bmatrix} \begin{bmatrix} \mathbf{C} & -\overline{\mathbf{D}} \\ \mathbf{D} & \overline{\mathbf{C}} \end{bmatrix} = \begin{bmatrix} \mathbf{AC} - \overline{\mathbf{B}}\mathbf{D} & -\overline{\mathbf{A}}\mathbf{D} - \overline{\mathbf{BC}} \\ \mathbf{BC} + \overline{\mathbf{A}}\mathbf{D} & -\mathbf{B}\overline{\mathbf{D}} + \overline{\mathbf{AC}} \end{bmatrix}.$$

Thus $\psi(\mathbf{MN}) = \psi(\mathbf{M})\psi(\mathbf{N})$ holds. □

Proposition 4.2. *If $m = n$, then ψ is an injective \mathbb{R} -algebra homomorphism.*

Proof. By Lemma 4.1, ψ is an \mathbb{R} -algebra homomorphism. Injectivity of ψ is clear. □

As in [1], one can characterize the image of ψ in $\text{Mat}(2n, \mathbb{C})$ as follows.

Lemma 4.3. *Let \mathbf{J} be the $2n \times 2n$ matrix defined as follows:*

$$\mathbf{J} = \begin{bmatrix} \mathbf{0} & -\mathbf{I}_n \\ \mathbf{I}_n & \mathbf{0} \end{bmatrix}.$$

Then,

$$\psi(\text{Mat}(n, \mathbb{H})) = \{\mathbf{N} \in \text{Mat}(2n, \mathbb{C}) \mid \mathbf{JN} = \overline{\mathbf{N}}\mathbf{J}\},$$

where $\overline{\mathbf{N}}$ denotes the complex conjugate of \mathbf{N} .

Proof. For the proof, see [1]. □

In [15], Study defined a determinant of a $n \times n$ quaternionic matrix which we denote by Sdet as follows:

$$\text{Sdet}(\mathbf{M}) = \det(\psi(\mathbf{M})),$$

where \det is the ordinary determinant. Sdet is called the *Study determinant*.

Sdet can be extended to the one for matrices whose entries are formal power series with coefficients in \mathbb{H} in the following way. Let $\mathbb{H}[[t]]$ be the ring of formal power series with coefficients in \mathbb{H} . We notice that t is a commuting indeterminate for \mathbb{H} , that is, $th = ht$ for any $h \in \mathbb{H}$. For $\alpha = \sum_{k \geq 0} \alpha_k t^k \in \mathbb{H}[[t]]$, the *conjugate* of α is defined by $\alpha^* = \sum_{k \geq 0} \alpha_k^* t^k$. Furthermore, let $\alpha_k = \alpha_k^S + j\alpha_k^P$ be the symplectic decomposition of α_k . Then the symplectic decomposition of α is defined by

$$\alpha = \alpha^S + j\alpha^P = \sum_{k \geq 0} \alpha_k^S t^k + j \sum_{k \geq 0} \alpha_k^P t^k.$$

One can extend ψ to the injective homomorphism ψ_t of \mathbb{R} -algebras from $\text{Mat}(n, \mathbb{H})[[t]]$ to $\text{Mat}(2n, \mathbb{C})[[t]]$ by defining $\psi_t(t) = t$. On the other hand, since $\text{Mat}(2n, \mathbb{C})[[t]] = \text{Mat}(2n, \mathbb{C}[[t]])$, one can also extend the determinant $\det : \text{Mat}(2n, \mathbb{C}) \rightarrow \mathbb{C}$ to $\det_t : \text{Mat}(2n, \mathbb{C})[[t]] \rightarrow \mathbb{C}[[t]]$ in the natural manner since \det is a polynomial of entries of a matrix. Both ψ_t and \det_t are continuous, thus

$$\det_t \cdot \psi_t : \text{Mat}(n, \mathbb{H})[[t]] \rightarrow \mathbb{C}[[t]]$$

is also continuous with respect to the topology of formal power series. We call $\det_t \cdot \psi_t$ the *Study determinant* for $\text{Mat}(n, \mathbb{H}[[t]])$ and denote by Sdet_t . We notice that the restriction of Sdet_t to $\text{Mat}(n, \mathbb{H})$ yields Sdet .

Before stating properties of Sdet_t , we mention some useful facts.

Lemma 4.4. *If $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are complex square matrices with same size. Suppose that \mathbf{A} is invertible and $\mathbf{AC} = \mathbf{CA}$, then*

$$\det \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \det(\mathbf{AD} - \mathbf{CB}).$$

Proof. For the proof, see for example [20]. □

Remark 4.5. *We notice that the Lemma 4.4 can be extended to matrices whose entries belong to $\mathbb{C}[[t]]$ immediately.*

Lemma 4.6.

$$\psi_t(\text{Mat}(n, \mathbb{H}[[t]])) = \{\mathbf{N} \in \text{Mat}(2n, \mathbb{C}[[t]]) \mid \mathbf{JN} = \overline{\mathbf{N}}\mathbf{J}\}.$$

Proof. By Lemma 4.3, it follows that

$$\begin{aligned} \exists \mathbf{M} &= \sum_{k \geq 0} \mathbf{M}_k t^k \in \text{Mat}(n, \mathbb{H}[[t]]), \quad \psi_t(\mathbf{M}) = \mathbf{N} = \sum_{k \geq 0} \mathbf{N}_k t^k \\ \Leftrightarrow \psi(\mathbf{M}_k) &= \mathbf{N}_k \quad \text{for } \forall k \geq 0 \Leftrightarrow \mathbf{JN}_k = \overline{\mathbf{N}}_k \mathbf{J} \quad \text{for } \forall k \geq 0 \\ \Leftrightarrow \mathbf{JN} &= \overline{\mathbf{N}}\mathbf{J}. \end{aligned}$$

Hence, the assertion holds. □

Lemma 4.7. *Let $\alpha = \sum_{k \geq 0} \alpha_k t^k \in \mathbb{H}[[t]]$, and $\alpha_k = \alpha_k^S + j\alpha_k^P$ the symplectic decomposition. Then,*

$$\alpha\alpha^* = \alpha^*\alpha = \sum_{m \geq 0} \sum_{k+\ell=m} (\alpha_k^S \overline{\alpha_\ell^S} + \alpha_k^P \overline{\alpha_\ell^P}) t^m = \alpha^S \overline{\alpha^S} + \alpha^P \overline{\alpha^P} \in \mathbb{C}[[t]].$$

Proof. By a direct calculation, we have:

$$\begin{aligned} \alpha\alpha^* &= \left(\sum_{k \geq 0} (\alpha_k^S + j\alpha_k^P) t^k \right) \left(\sum_{\ell \geq 0} (\alpha_\ell^S + j\alpha_\ell^P)^* t^\ell \right) \\ &= \left(\sum_{k \geq 0} (\alpha_k^S + j\alpha_k^P) t^k \right) \left(\sum_{\ell \geq 0} (\overline{\alpha_\ell^S} - \overline{\alpha_\ell^P} j) t^\ell \right) \\ &= \sum_{m \geq 0} \sum_{k+\ell=m} \left\{ (\alpha_k^S \overline{\alpha_\ell^S} + \overline{\alpha_k^P} \alpha_\ell^P) - j(\overline{\alpha_k^S} \alpha_\ell^P - \alpha_k^P \overline{\alpha_\ell^S}) \right\} t^m \\ &= \sum_{m \geq 0} \sum_{k+\ell=m} (\alpha_k^S \overline{\alpha_\ell^S} + \overline{\alpha_k^P} \alpha_\ell^P) t^m \\ &= \alpha^S \overline{\alpha^S} + \alpha^P \overline{\alpha^P}. \end{aligned}$$

Furthermore, $\alpha^S \overline{\alpha^S} + \alpha^P \overline{\alpha^P}$ belongs to $\mathbb{C}[[t]]$. Similarly, the same conclusion holds for $\alpha^* \alpha$. \square

Lemma 4.7 immediately implies:

Corollary 4.8. *For all $\alpha, \beta \in \mathbb{H}[[t]]$, $\alpha\alpha^*\beta\beta^* = \beta\beta^*\alpha\alpha^*$.*

Proposition 4.9. (i) $\text{Sdet}_t(\mathbf{M}) \in \mathbb{R}[[t]]$ for $\mathbf{M} \in \text{Mat}(n, \mathbb{H}[[t]])$

(ii) *The constant term of $\text{Sdet}_t(\mathbf{M})$ is equal to 0 \Leftrightarrow \mathbf{M} is singular.*

(iii) $\text{Sdet}_t(\mathbf{MN}) = \text{Sdet}_t(\mathbf{M}) \text{Sdet}_t(\mathbf{N})$ for $\mathbf{M}, \mathbf{N} \in \text{Mat}(n, \mathbb{H}[[t]])$.

(iv) *If \mathbf{N} is obtained from \mathbf{M} by adding a left-multiple of a row to another row or a right-multiple of a column to another column, then $\text{Sdet}_t(\mathbf{N}) = \text{Sdet}_t(\mathbf{M})$.*

(v) *If \mathbf{N} is obtained from \mathbf{M} by exchanging a row with another row or a column with another column, then $\text{Sdet}_t(\mathbf{N}) = \text{Sdet}_t(\mathbf{M})$.*

(vi) $\text{Sdet}_t(\alpha\mathbf{M}) = \text{Sdet}_t(\mathbf{M}\alpha) = (\alpha\alpha^*)^n \text{Sdet}_t(\mathbf{M})$ for $\mathbf{M} \in \text{Mat}(n, \mathbb{H}[[t]])$ and $\alpha \in \mathbb{H}[[t]]$.

(vii) *If $\mathbf{M} \in \text{Mat}(n, \mathbb{H}[[t]])$ is of the form:*

$$\mathbf{M} = \begin{bmatrix} \lambda_1 & * & \cdots & * \\ 0 & \lambda_2 & & * \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ * & \lambda_2 & & 0 \\ \vdots & & \ddots & \vdots \\ * & * & \cdots & \lambda_n \end{bmatrix},$$

Then, $\text{Sdet}_t(\mathbf{M}) = \prod_{i=1}^n \lambda_i \lambda_i^$.*

(viii) Let \mathbf{A} be an $m \times n$ matrix and \mathbf{B} an $n \times m$ matrix. Then

$$\text{Sdet}_t(\mathbf{I}_m - \mathbf{A}\mathbf{B}) = \text{Sdet}_t(\mathbf{I}_n - \mathbf{B}\mathbf{A}).$$

Proof. (i) It follows from Lemma 4.6 that

$$\det_t(\mathbf{N}) = \det_t(\mathbf{J}^{-1}\overline{\mathbf{N}}\mathbf{J}) = \det_t(\overline{\mathbf{N}}) = \overline{\det_t(\mathbf{N})},$$

for any $\mathbf{N} \in \psi_t(\text{Mat}(n, \mathbb{H}[[t]]))$. Thus $\text{Sdet}_t(\mathbf{M})$ must belong to $\mathbb{R}[[t]]$.

(ii) It is known that a matrix over a commutative ring is invertible if and only if its determinant is invertible, and that a formal power series is invertible if and only if its constant term is invertible. Hence, for any $\mathbf{M} \in \text{Mat}(n, \mathbb{H}[[t]])$, the constant term of $\det_t \cdot \psi_t(\mathbf{M})$ is equal to 0 $\Leftrightarrow \det_t \cdot \psi_t(\mathbf{M})$ is not invertible $\Leftrightarrow \psi_t(\mathbf{M})$ is singular. Taking conjugate and inverse of $\psi_t(\mathbf{M})\mathbf{J} = \mathbf{J}\psi_t(\mathbf{M})$, we have:

$$\mathbf{J}\psi_t(\mathbf{M})^{-1} = \overline{\psi_t(\mathbf{M})^{-1}\mathbf{J}}.$$

Hence by Lemma 4.6, $\psi_t(\mathbf{M})^{-1} \in \psi_t(\text{Mat}(n, \mathbb{H}[[t]]))$ and it follows that $\psi_t(\mathbf{M})$ is singular $\Leftrightarrow \mathbf{M}$ is singular.

(iii) Since ψ_t is an algebra homomorphism, (iii) immediately holds.

(iv) Let $b = b_1 + jb_2 \in \mathbb{H}[[t]]$ ($b_1, b_2 \in \mathbb{C}[[t]]$). It suffices to show that for $k \neq \ell$, the $n \times n$ matrix $\mathbf{B}_{k\ell}(b) = \mathbf{I}_n + b\mathbf{E}_{k\ell}$ satisfies $\text{Sdet}_t(\mathbf{B}_{k\ell}(b)) = 1$. Since four blocks of

$$\psi_t(\mathbf{B}_{k\ell}(b)) = \begin{bmatrix} \mathbf{I}_n + b_1\mathbf{E}_{k\ell} & -\overline{b_2}\mathbf{E}_{k\ell} \\ b_2\mathbf{E}_{k\ell} & \mathbf{I}_n + \overline{b_1}\mathbf{E}_{k\ell} \end{bmatrix}$$

commute with each other, we can apply Remark 4.5 and obtain:

$$\det_t \cdot \psi_t(\mathbf{B}_{k\ell}(b)) = \det_t((\mathbf{I}_n + b_1\mathbf{E}_{k\ell})(\mathbf{I}_n + \overline{b_1}\mathbf{E}_{k\ell}) + b_2\mathbf{E}_{k\ell}\overline{b_2}\mathbf{E}_{k\ell}) = 1.$$

(v) We prove for row exchange. Putting $\mathbf{N} = \mathbf{P}\mathbf{M}$ for the permutation matrix \mathbf{P} corresponding to some transposition, it follows that:

$$\begin{aligned} \text{Sdet}_t(\mathbf{N}) &= \text{Sdet}_t(\mathbf{P}\mathbf{M}) = \det_t \left(\begin{bmatrix} \mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{P} \end{bmatrix} \psi_t(\mathbf{M}) \right) \\ &= \det_t \left(\begin{bmatrix} \mathbf{P} & \mathbf{O} \\ \mathbf{O} & \mathbf{P} \end{bmatrix} \right) \det_t(\psi_t(\mathbf{M})) = \det(\mathbf{P})^2 \text{Sdet}_t(\mathbf{M}) \\ &= \text{Sdet}_t(\mathbf{M}). \end{aligned}$$

(vi) Let $\mathbf{M} \in \text{Mat}(n, \mathbb{H}[[t]])$ and $\alpha = \alpha^S + j\alpha^P \in \mathbb{H}[[t]]$ ($\alpha^S, \alpha^P \in \mathbb{C}[[t]]$) be the symplectic decomposition. Then by using Lemma 4.1, Remark 4.5 and Lemma 4.7, we have:

$$\begin{aligned} \text{Sdet}_t(\alpha\mathbf{M}) &= \det_t(\psi_t(\alpha\mathbf{M})) = \det_t(\psi_t(\alpha\mathbf{I}_n)\psi_t(\mathbf{M})) \\ &= \det_t \begin{bmatrix} \alpha^S\mathbf{I}_n & -\overline{\alpha^P}\mathbf{I}_n \\ \alpha^P\mathbf{I}_n & \overline{\alpha^S}\mathbf{I}_n \end{bmatrix} \det_t(\psi_t(\mathbf{M})) \\ &= \det_t((\alpha^S\mathbf{I}_n)(\overline{\alpha^S}\mathbf{I}_n) + (\alpha^P\mathbf{I}_n)(\overline{\alpha^P}\mathbf{I}_n)) \text{Sdet}_t(\mathbf{M}) \\ &= \det_t((\alpha^S\overline{\alpha^S} + \alpha^P\overline{\alpha^P})\mathbf{I}_n) \text{Sdet}_t(\mathbf{M}) \\ &= (\alpha\alpha^*)^n \text{Sdet}_t(\mathbf{M}). \end{aligned}$$

In the same way, we can deduce $\text{Sdet}_t(\mathbf{M}\alpha) = (\alpha\alpha^*)^n \text{Sdet}_t(\mathbf{M})$.

(vii) For a $2n \times 2n$ matrix \mathbf{N} and any two subsets $I = \{i_1, i_2, \dots, i_r\}$, $J = \{j_1, j_2, \dots, j_s\}$ of $[2n]$, \mathbf{N}^{IJ} denotes the submatrix obtained from \mathbf{N} by deleting i_1, i_2, \dots, i_r th rows and j_1, j_2, \dots, j_s th columns. Then by definitions of Sdet_t and ψ_t , we get the following:

$$\begin{aligned}
\text{Sdet}_t(\mathbf{M}) &= \det_t(\psi_t(\mathbf{M})) = \det_t \begin{bmatrix} \mathbf{M}^S & -\overline{\mathbf{M}^P} \\ \mathbf{M}^P & \overline{\mathbf{M}^S} \end{bmatrix} \\
&= \det_t \begin{bmatrix} \lambda_1^S & * & \cdots & * & -\overline{\lambda_1^P} & * & \cdots & * \\ 0 & \lambda_2^S & & * & 0 & -\overline{\lambda_2^P} & & * \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^S & 0 & 0 & \cdots & -\overline{\lambda_n^P} \\ \lambda_1^P & * & \cdots & * & \overline{\lambda_1^S} & * & \cdots & * \\ 0 & \lambda_2^P & & * & 0 & \overline{\lambda_2^S} & & * \\ \vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^P & 0 & 0 & \cdots & \overline{\lambda_n^S} \end{bmatrix} \\
&= \lambda_1^S \psi_t(\mathbf{M})^{\{1\}\{1\}} + (-1)^{n+2} \lambda_1^P \psi_t(\mathbf{M})^{\{n+1\}\{1\}} \\
&= \lambda_1^S \overline{\lambda_1^S} \psi_t(\mathbf{M})^{\{1,n+1\}\{1,n+1\}} + \lambda_1^P \overline{\lambda_1^P} \psi_t(\mathbf{M})^{\{1,n+1\}\{1,n+1\}} \\
&= (\lambda_1^S \overline{\lambda_1^S} + \lambda_1^P \overline{\lambda_1^P}) \psi_t(\mathbf{M})^{\{1,n+1\}\{1,n+1\}} \\
&= \lambda_1 \lambda_1^* \psi_t(\mathbf{M})^{\{1,n+1\}\{1,n+1\}} \\
&= \lambda_1 \lambda_1^* \lambda_2 \lambda_2^* \psi_t(\mathbf{M})^{\{1,2,n+1,n+2\}\{1,2,n+1,n+2\}} = \dots \\
&= \prod_{i=1}^n \lambda_i \lambda_i^*
\end{aligned}$$

(viii) Let \mathbf{A} be an $m \times n$ matrix and \mathbf{B} an $n \times m$ matrix. Then, by the definition of the Study determinant, we have:

$$\begin{aligned}
\text{Sdet}_t(\mathbf{I}_m - \mathbf{AB}) &= \det_t(\psi_t(\mathbf{I}_m - \mathbf{AB})) = \det_t(\mathbf{I}_{2m} - \psi_t(\mathbf{A})\psi_t(\mathbf{B})) \\
&= \det_t(\mathbf{I}_{2n} - \psi_t(\mathbf{B})\psi_t(\mathbf{A})) = \det_t(\psi_t(\mathbf{I}_n - \mathbf{BA})) \\
&= \text{Sdet}_t(\mathbf{I}_n - \mathbf{BA}).
\end{aligned}$$

□

Remark 4.10. Sdet_t is not multilinear as \det_t is. Furthermore, $\text{Sdet}_t({}^T\mathbf{M}) = \text{Sdet}_t(\mathbf{M})$ does not hold in general where ${}^T\mathbf{M}$ is the transpose of \mathbf{M} .

5 The Ihara zeta function of a graph

Let $G = (V(G), E(G))$ be a finite connected graph with the set $V(G)$ of vertices and the set $E(G)$ of unoriented edges uv joining two vertices u and v . We assume that G has neither loops nor multiple edges throughout. For $uv \in E(G)$, an arc (u, v) is the oriented edge from u to v . Let $D(G) = \{(u, v), (v, u) \mid uv \in E(G)\}$ and $|V(G)| = n$, $|E(G)| = m$, $|D(G)| = 2m$.

For $e = (u, v) \in D(G)$, $o(e) = u$ denotes the *origin* and $t(e) = v$ the *terminal* of e respectively. Furthermore, let $e^{-1} = (v, u)$ be the *inverse* of $e = (u, v)$. The *degree* $\deg v = \deg_G v$ of a vertex v of G is the number of edges incident to v . For a natural number k , a graph G is called *k-regular* if $\deg_G v = k$ for each vertex v of G . A *path* P of length ℓ in G is a sequence $P = (e_1, \dots, e_\ell)$ of ℓ arcs such that $e_i \in D(G)$ and $t(e_i) = o(e_{i+1})$ for $i \in \{1, \dots, \ell - 1\}$. We set $o(P) = o(e_1)$ and $t(P) = t(e_\ell)$. $|P|$ denotes the length of P . We say that a path $P = (e_1, \dots, e_\ell)$ has a *backtracking* if $e_{i+1} = e_i^{-1}$ for some $i (1 \leq i \leq \ell - 1)$. A path P is said to be a *cycle* if $t(P) = o(P)$. The *inverse* of a path $P = (e_1, \dots, e_\ell)$ is the path $(e_\ell^{-1}, \dots, e_1^{-1})$ and is denoted by P^{-1} .

An equivalence relation between cycles is given as follows. Two cycles $C_1 = (e_1, \dots, e_\ell)$ and $C_2 = (f_1, \dots, f_\ell)$ are said to be *equivalent* if there exists k such that $f_j = e_{j+k}$ for all j where indices are treated modulo ℓ . Let $[C]$ be the equivalence class which contains the cycle C . Let B^r be the cycle obtained by going r times around a cycle B . Such a cycle is called a *power* of B . A cycle C is said to be *reduced* if both C and C^2 have no backtracking. Furthermore, a cycle C is said to be *prime* if it is not a power of a strictly smaller cycle. Note that each equivalence class of prime, reduced cycles of a graph G corresponds to a unique conjugacy class of the fundamental group $\pi_1(G, v)$ of G at a vertex v of G .

The *Ihara zeta function* of a graph G is a function of $t \in \mathbf{C}$ with $|t|$ sufficiently small, defined by

$$\mathbf{Z}(G, t) = \mathbf{Z}_G(t) = \prod_{[C]} (1 - t^{|C|})^{-1},$$

where $[C]$ runs over all equivalence classes of prime, reduced cycles of G .

Let $\mathbf{B} = (\mathbf{B}_{ef})_{e, f \in D(G)}$ and $\mathbf{J}_0 = (\mathbf{J}_{ef})_{e, f \in D(G)}$ be $2m \times 2m$ matrices defined as follows:

$$\mathbf{B}_{ef} = \begin{cases} 1 & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{J}_{ef} = \begin{cases} 1 & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the matrix $\mathbf{B} - \mathbf{J}_0$ is called the *edge matrix* of G .

Theorem 5.1 (Hashimoto[6]; Bass[2]). *Let G be a finite connected graph. Then the reciprocal of the Ihara zeta function of G is given by*

$$\mathbf{Z}(G, t)^{-1} = \det(\mathbf{I} - t(\mathbf{B} - \mathbf{J}_0)) = (1 - t^2)^{r-1} \det(\mathbf{I} - t\mathbf{A} + t^2(\mathbf{D} - \mathbf{I})),$$

where r and \mathbf{A} are the Betti number and the adjacency matrix of G , respectively, and $\mathbf{D} = (d_{ij})$ is the diagonal matrix with $d_{ii} = \deg v_i$, $V(G) = \{v_1, \dots, v_n\}$. $\mathbf{Z}(G, t)$ has an expression as the exponential of a generating function as follows:

$$\mathbf{Z}(G, t) = \exp \left(\sum_{k \geq 1} \frac{N_k}{k} t^k \right),$$

where N_k is the number of equivalence classes of reduced cycles of length k .

The matrix \mathbf{D} is called the *degree matrix* of G .

A weighted zeta function of a graph was first defined in Hashimoto [6] by giving a weight for each edge of a graph. After that, it was generalized in Stark and Terras [14] by giving a weight for each arc of a graph as follows. Let $D(G) = \{e_1, \dots, e_m, e_{m+1}, \dots, e_{2m}\}$ where

$e_{m+i} = e_i^{-1}$ for $1 \leq i \leq m$. For each arc e_i ($i = 1, \dots, 2m$), associate a complex number u_i , and set $\mathbf{u} = (u_1, \dots, u_{2m})$. Let $g(C) = u_{i_1} \cdots u_{i_k}$ for a cycle $C = (e_{i_1}, \dots, e_{i_k})$. Then the edge zeta function $\zeta_G(\mathbf{u})$ of G is defined by

$$\zeta_G(\mathbf{u}) = \prod_{[C]} (1 - g(C))^{-1},$$

where $[C]$ runs over all equivalent classes of prime, reduced cycles of G .

Theorem 5.2 (Stark and Terras[14]). *Let G be a connected graph with m edges. Then*

$$\zeta_G(\mathbf{u})^{-1} = \det(\mathbf{I}_{2m} - (\mathbf{B} - \mathbf{J}_0)\mathbf{U}) = \det(\mathbf{I}_{2m} - \mathbf{U}(\mathbf{B} - \mathbf{J}_0)),$$

where $\mathbf{U} = \text{diag}(u_1, \dots, u_{2m})$ is the diagonal matrix.

In [14], the multivariable analogue $N_k(\mathbf{u})$ of N_k was defined and used to obtain the determinant expression. Mizuno and Sato [11] defined the weighted zeta function of a graph G , and gave a determinant expression by the weighted matrix of G . Later, they called this zeta function the first weighted zeta function in order to distinguish it from another zeta function which Sato defined after. Assigning a complex number $w(e)$ to each arc $e \in D(G)$, we define $\mathbf{W} = (w_{uv})_{u,v \in V(G)}$ to be the $n \times n$ matrix as follows:

$$w_{uv} = \begin{cases} w(e) & \text{if } e = (u, v) \in D(G), \\ 0 & \text{otherwise,} \end{cases}$$

\mathbf{W} is called the *weighted matrix* of G . For each path $P = (e_{i_1}, e_{i_2}, \dots, e_{i_d})$, the *norm* $w(P)$ of P is defined by $w(P) = w(e_{i_1})w(e_{i_2}) \cdots w(e_{i_d})$. The *(first) weighted zeta function* of G is defined by

$$\mathbf{Z}(G, w, t) = \prod_{[C]} (1 - w(C)t^{|C|})^{-1}, \quad (5.14)$$

where $[C]$ runs over all equivalent classes of prime, reduced cycles of G .

Theorem 5.3 (Mizuno and Sato[11]). *Let G be a connected graph with n vertices. Suppose that $w(e^{-1}) = w(e)^{-1}$ for $e \in D(G)$. Then*

$$\mathbf{Z}(G, w, t)^{-1} = (1 - t^2)^{m-n} \det(\mathbf{I}_n - t\mathbf{W} + t^2(\mathbf{D} - \mathbf{I}_n)).$$

$\mathbf{Z}(G, w, t)$ has an expression as the exponential of a generating function as follows:

$$\mathbf{Z}(G, w, t) = \exp \left(\sum_{[C]} \sum_{s \geq 1} \frac{1}{s} w(C)^s t^{|C|s} \right),$$

where $[C]$ runs over all equivalent classes of prime, reduced cycles of G .

Later, Watanabe and Fukumizu [18] gave a determinant expression for the edge zeta function of a graph with no additional condition $w(e^{-1}) = w(e)^{-1}$. Rearranging arcs so that $D(G) = \{e_1, e_1^{-1}, \dots, e_m, e_m^{-1}\}$, and putting $u_i = w(e_i)$, \mathbf{U} turns out to be

$$\mathbf{U} = \begin{bmatrix} w(e_1) & & & & 0 \\ & w(e_1^{-1}) & & & \\ & & \ddots & & \\ & & & w(e_m) & \\ 0 & & & & w(e_m^{-1}) \end{bmatrix}.$$

In [18], two $n \times n$ matrices $\hat{\mathbf{A}} = (\hat{\mathbf{A}}_{uv})_{u,v \in V(G)}$ and $\hat{\mathbf{D}} = (\hat{\mathbf{D}}_{uv})_{u,v \in V(G)}$ were defined as follows:

$$\hat{\mathbf{A}}_{uv} = \begin{cases} \frac{w(u,v)}{1 - w(u,v)w(v,u)} & \text{if } (u,v) \in D(G), \\ 0 & \text{otherwise,} \end{cases}$$

$$\hat{\mathbf{D}}_{uv} = \delta_{uv} \sum_{\substack{e \in D(G) \\ o(e)=u}} \frac{w(e)w(e^{-1})}{1 - w(e)w(e^{-1})}.$$

where δ_{uv} denotes the Kronecker's delta. They derived their determinant expression with $\hat{\mathbf{A}}$ and $\hat{\mathbf{D}}$ instead of the weighted matrix and the degree matrix of G .

Theorem 5.4 (Watanabe and Fukumizu[18]).

$$\zeta_G(\mathbf{u})^{-1} = \det(\mathbf{I}_{2m} - \mathbf{U}(\mathbf{B} - \mathbf{J}_0)) = \det(\mathbf{I}_n + \hat{\mathbf{D}} - \hat{\mathbf{A}}) \prod_{i=1}^m (1 - w(e_i)w(e_i^{-1}))$$

6 The quaternionic weighted zeta function of a graph

We continue with the notations in former sections. We shall define the quaternionic weighted zeta function of a graph. First we extend $w(e)$ to a quaternion. Let w be a map from $V(G) \times V(G)$ to \mathbb{H} such that $w(u,v) = 0$ if $(u,v) \notin D(G)$. We may write $w(e)$ instead of $w(u,v)$ if $e = (u,v) \in D(G)$. We call $w(e)$ the *quaternionic weight* of $e \in D(G)$. For a path $P = (e_1, \dots, e_\ell)$ of G , the *norm* (or the *quaternionic weight*) $w(P)$ of P is defined by $w(P) = w(e_1)w(e_2) \cdots w(e_\ell)$. We arrange the arcs e_1, e_2, \dots, e_{2m} such that $e_{m+k} = e_k^{-1}$ for $k = 1, \dots, m$. Now, we define the *quaternionic weighted zeta function* of G to be the element of $\mathbb{H}[[t]]$ as follows:

$$\mathbf{Z}_{\mathbb{H}}(G, w, t) = \prod_C \left\{ (1 - w(C)t^{|C|})(1 - w(C)t^{|C|})^* \right\}^{-1}, \quad (6.15)$$

where C runs over all reduced cycles $C = e_{i_1}e_{i_2} \cdots e_{i_r}$ such that $i_1i_2 \cdots i_r \in L_{[2m]}$. If $w(e) = 1$ for any $e \in D(G)$, then the quaternionic weighted zeta function of G is reduced to the square of the Ihara zeta function of G .

Remark 6.1. (i) *The primeness is not required explicitly since $i_1i_2 \cdots i_r \in L_{[2m]}$ implies the primeness of $e_{i_1}e_{i_2} \cdots e_{i_r}$.*

(ii) (6.15) *does not depend on the order in which inverses of $(1 - w(C)t^{|C|})(1 - w(C)t^{|C|})^*$ are multiplied since $(1 - w(C)t^{|C|})(1 - w(C)t^{|C|})^* \in \mathbb{C}[[t]]$ by Lemma 4.7. Indeed, the inverses are in $\mathbb{R}[[t]]$ since $(1 - w(C)t^{|C|})(1 - w(C)t^{|C|})^* = 1 - 2\{\operatorname{Re} w(C)\}t^{|C|} + |w(C)|^2t^{2|C|}$.*

We shall show that $\mathbf{Z}_{\mathbb{H}}(G, w, t)$ can be expressed as the exponential of a generating function. We recall that Theorem 3.1, Proposition 3.2, Proposition 3.3 and Corollary 3.4 hold even if we replace $\mathbb{Q}[X^*][[t]]$ with $\mathbb{H}[[t]]$. By Remark 6.1, we have:

$$\mathbf{Z}_{\mathbb{H}}(G, w, t) = \prod_C \left\{ 1 - 2\{\operatorname{Re} w(C)\}t^{|C|} + |w(C)|^2t^{2|C|} \right\}^{-1}.$$

Since $\alpha_C = 2\{\operatorname{Re} w(C)\}t^{|C|} - |w(C)|^2 t^{2|C|}$ is an element of $\mathbb{R}[[t]]$ and has zero constant term,

$$\{1 - 2\{\operatorname{Re} w(C)\}t^{|C|} + |w(C)|^2 t^{2|C|}\}^{-1} = \{1 - \alpha_C\}^{-1} = 1 + \sum_{k \geq 1} \alpha_C^k$$

is also an element of $\mathbb{R}[[t]]$ and $\beta_C = \sum_{k \geq 1} \alpha_C^k$ has zero constant term. Since the constant term of $\mathbf{Z}_{\mathbb{H}}(G, w, t)$ is equal to 1, we can derive the following:

$$\begin{aligned} \log \mathbf{Z}_{\mathbb{H}}(G, w, t) &= \log \prod_C (1 + \beta_C) = \sum_C \log(1 + \beta_C) \\ &= \sum_C \log((1 - \alpha_C)^{-1}) = - \sum_C \log(1 - \alpha_C) \\ &= - \sum_C \log \left\{ (1 - w(C)t^{|C|})(1 - w(C)^* t^{|C|}) \right\} \\ &= - \sum_C \left\{ \log(1 - w(C)t^{|C|}) + \log(1 - w(C)^* t^{|C|}) \right\} \\ &= - \sum_C \sum_{n \geq 1} \frac{-1}{n} \{w(C)^n + (w(C)^*)^n\} t^{n|C|} \\ &= \sum_C \sum_{n \geq 1} \frac{2\operatorname{Re}(w(C)^n)}{n} t^{n|C|} = \sum_{n \geq 1} \sum_C \frac{2\operatorname{Re}(w(C)^n)}{n} t^{n|C|}. \end{aligned}$$

Therefore $\log \mathbf{Z}_{\mathbb{H}}(G, w, t)$ has zero constant term, and it follows that:

Theorem 6.2. $\mathbf{Z}_{\mathbb{H}}(G, w, t)$ has the expression as the exponential of a generating function as follows:

$$\mathbf{Z}_{\mathbb{H}}(G, w, t) = \exp \left(\sum_{n \geq 1} \sum_C \frac{2\operatorname{Re}(w(C)^n)}{n} t^{n|C|} \right).$$

where C runs over all reduced cycles $C = e_{i_1} e_{i_2} \cdots e_{i_r}$ such that $i_1 i_2 \cdots i_r \in L_{[2m]}$.

We can also express $\mathbf{Z}_{\mathbb{H}}(G, w, t)$ as the square of the exponential of a generating function as follows:

$$\mathbf{Z}_{\mathbb{H}}(G, w, t) = \left\{ \exp \left(\sum_{n \geq 1} \sum_C \frac{\operatorname{Re}(w(C)^n)}{n} t^{n|C|} \right) \right\}^2.$$

7 Study determinant expressions for $\mathbf{Z}_{\mathbb{H}}(G, w, t)$

Consider an $n \times n$ quaternionic matrix $\mathbf{W} = (\mathbf{W}_{uv})_{u,v \in V(G)} \in \operatorname{Mat}(n, \mathbb{H})$ with (u, v) -entry equals 0 if $(u, v) \notin D(G)$. We call \mathbf{W} a *quaternionic weighted matrix* of G . Furthermore, let $w(u, v) = \mathbf{W}_{uv}$ for $u, v \in V(G)$ and $w(e) = w(u, v)$ if $e = (u, v) \in D(G)$.

For a quaternionic weighted matrix \mathbf{W} of G , we define two $2m \times 2m$ matrices $\mathbf{B}_w = (\mathbf{B}_{ef}^{(w)})_{e,f \in D(G)}$ and $\mathbf{J}_w = (\mathbf{J}_{ef}^{(w)})_{e,f \in D(G)}$ as follows:

$$\mathbf{B}_{ef}^{(w)} = \begin{cases} w(e) & \text{if } t(e) = o(f), \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{J}_{ef}^{(w)} = \begin{cases} w(e) & \text{if } f = e^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

We call $\mathbf{B}_w - \mathbf{J}_w$ the *quaternionic weighted edge matrix*. Then the Study determinant expression of $\mathbf{Z}_{\mathbb{H}}(G, w, t)$ of Hashimoto type by use of the quaternionic weighted edge matrix is stated as follows:

Theorem 7.1.

$$\mathbf{Z}_{\mathbb{H}}(G, w, t)^{-1} = \text{Sdet}_t(\mathbf{I}_{2m} - (\mathbf{B}_w - \mathbf{J}_w)t).$$

Beforehand, we mention that the right hand side does not depend on the way of arranging arcs which determines row and column indexes. Indeed, if we rearrange arcs, then it turns into $\text{Sdet}_t(\mathbf{P}(\mathbf{I}_{2m} - (\mathbf{B}_w - \mathbf{J}_w)t)\mathbf{P}^{-1})$ for some permutation matrix \mathbf{P} . However, from Proposition 4.9 (v), this is equal to $\text{Sdet}_t(\mathbf{I}_{2m} - (\mathbf{B}_w - \mathbf{J}_w)t)$.

Proof. Set $A = \mathbb{H}$ in Proposition 2.3. Then we can take Study determinants of both sides in (2.7), so that

$$\begin{aligned} & \text{Sdet}_t(\mathbf{I}_{2m} - \mathbf{A}t) \\ &= \text{Sdet}_t \left(\prod_{\substack{(i_1, j_1) \cdots (i_r, j_r) \in L_{[2m] \times [2m]} \\ j_k = i_{k+1} \ (k=1, \dots, r-1)}}^{<} (\mathbf{I}_n - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{r-1} i_r} a_{i_r j_r} \mathbf{E}_{i_1 j_r} t^r) \right) \\ &= \prod_{\substack{(i_1, j_1) \cdots (i_r, j_r) \in L_{[2m] \times [2m]} \\ j_k = i_{k+1} \ (k=1, \dots, r-1)}} \text{Sdet}_t(\mathbf{I}_n - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_{r-1} i_r} a_{i_r j_r} \mathbf{E}_{i_1 j_r} t^r). \end{aligned} \tag{7.16}$$

We notice that the last formula does not depend on the order in which Sdet_t are multiplied since Sdet_t take values in $\mathbb{R}[[t]]$. It follows from Proposition 4.9 (vii) that if $j_r = i_1$, then

$$\begin{aligned} & \text{Sdet}_t(\mathbf{I}_n - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r i_1} \mathbf{E}_{i_1 i_1} t^r) \\ &= (1 - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r i_1} t^r)(1 - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r i_1} t^r)^*, \end{aligned}$$

and otherwise,

$$\text{Sdet}_t(\mathbf{I}_n - a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_r j_r} \mathbf{E}_{i_1 j_r} t^r) = 1.$$

Putting $\mathbf{A} = \mathbf{B}_w - \mathbf{J}_w$, then

$$a_{ij} = a_{e_i e_j} = \begin{cases} w(e_i) & \text{if } t(e_i) = o(e_j) \text{ and } e_j \neq e_i^{-1}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, (7.16) yields:

$$\begin{aligned} & \text{Sdet}_t(\mathbf{I}_{2m} - (\mathbf{B}_w - \mathbf{J}_w)t) \\ &= \prod_{\substack{(i_1, i_2) \cdots (i_r, i_1) \in L_{[2m] \times [2m]} \\ e_{i_1} e_{i_2} \cdots e_{i_r} : \text{reduced cycle}}} \text{Sdet}_t(\mathbf{I}_n - w(e_{i_1}) w(e_{i_2}) \cdots w(e_{i_r}) \mathbf{E}_{i_1 i_1} t^r) \\ &= \prod_{\substack{(i_1, i_2) \cdots (i_r, i_1) \in L_{[2m] \times [2m]} \\ e_{i_1} e_{i_2} \cdots e_{i_r} : \text{reduced cycle}}} (1 - w(e_{i_1}) \cdots w(e_{i_r}) t^r) (1 - w(e_{i_1}) \cdots w(e_{i_r}) t^r)^*. \end{aligned}$$

Each Lyndon word $(i_1, i_2) \cdots (i_r, i_1)$ in $L_{[2m] \times [2m]}$ corresponds to a Lyndon word $i_1 i_2 \cdots i_r$ in $L_{[2m]}$ bijectively. Hence we have:

$$\begin{aligned} & \text{Sdet}_t(\mathbf{I}_{2m} - (\mathbf{B}_w - \mathbf{J}_w)t) \\ &= \prod_{\substack{i_1 i_2 \cdots i_r \in L_{[2m]} \\ e_{i_1} e_{i_2} \cdots e_{i_r} : \text{reduced cycle}}} (1 - w(e_{i_1}) \cdots w(e_{i_r})t^r)(1 - w(e_{i_1}) \cdots w(e_{i_r})t^r)^*. \end{aligned}$$

Remark 6.1 (ii) implies the inverse of the right hand side is $\mathbf{Z}_{\mathbb{H}}(G, w, t)$. Hence we obtain the conclusion as desired. \square

Finally, we shall show the Study determinant expression of Bass type. We define $n \times n$ matrices $\tilde{\mathbf{W}} = (\tilde{\mathbf{W}}_{uv})_{u,v \in V(G)}$ and $\tilde{\mathbf{D}} = (\tilde{\mathbf{D}}_{uv})_{u,v \in V(G)}$ to be as follows:

$$\begin{aligned} \tilde{\mathbf{W}}_{uv} &= \begin{cases} (1 - w(e)w(e^{-1})t^2)^{-1}w(e) & \text{if } e = (u, v) \in D(G) \\ 0 & \text{otherwise,} \end{cases} \\ \tilde{\mathbf{D}}_{uv} &= \delta_{uv} \sum_{\substack{e \in D(G) \\ o(e)=u}} (1 - w(e)w(e^{-1})t^2)^{-1}w(e)w(e^{-1}). \end{aligned}$$

Then the Study determinant expression of Bass type is stated as follows:

Theorem 7.2.

$$\begin{aligned} & \mathbf{Z}_{\mathbb{H}}(G, w, t)^{-1} \\ &= \text{Sdet}_t(\mathbf{I}_n - t\tilde{\mathbf{W}} + t^2\tilde{\mathbf{D}}) \prod_{i=1}^m (1 - w(e_i)w(e_i^{-1})t^2)(1 - w(e_i)w(e_i^{-1})t^2)^*. \end{aligned}$$

Proof. We readily see that:

$$\begin{aligned} \text{Sdet}_t(\mathbf{I}_{2m} - t(\mathbf{B}_w - \mathbf{J}_w)) &= \text{Sdet}_t(\mathbf{I}_{2m} + t\mathbf{J}_w - t\mathbf{B}_w) \\ &= \text{Sdet}_t(\mathbf{I}_{2m} - t\mathbf{B}_w(\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1}) \\ &\quad \times \text{Sdet}_t(\mathbf{I}_{2m} + t\mathbf{J}_w). \end{aligned}$$

Now, let $\mathbf{K} = (\mathbf{K}_{ev})_{e \in D(G); v \in V(G)}$ and $\mathbf{L} = (\mathbf{L}_{ev})_{e \in D(G); v \in V(G)}$ be two $2m \times n$ matrices defined as follows:

$$\mathbf{K}_{ev} := \begin{cases} w(e) & \text{if } t(e) = v, \\ 0 & \text{otherwise,} \end{cases} \quad \mathbf{L}_{ev} := \begin{cases} 1 & \text{if } o(e) = v, \\ 0 & \text{otherwise.} \end{cases}$$

Then one can easily check that:

$$\mathbf{K}^T \mathbf{L} = \mathbf{B}_w.$$

Thus, from Proposition 4.9 (viii), we get the following:

$$\begin{aligned} \text{Sdet}_t(\mathbf{I}_{2m} - t(\mathbf{B}_w - \mathbf{J}_w)) &= \text{Sdet}_t(\mathbf{I}_{2m} - t\mathbf{K}^T \mathbf{L}(\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1}) \\ &\quad \times \text{Sdet}_t(\mathbf{I}_{2m} + t\mathbf{J}_w) \\ &= \text{Sdet}_t(\mathbf{I}_n - t^T \mathbf{L}(\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1} \mathbf{K}) \\ &\quad \times \text{Sdet}_t(\mathbf{I}_{2m} + t\mathbf{J}_w). \end{aligned} \tag{7.17}$$

Arranging arcs so that $D(G) = \{e_1, e_1^{-1}, \dots, e_m, e_m^{-1}\}$, we have from Proposition 4.9 (iv), (vii) that:

$$\begin{aligned}
\text{Sdet}_t(\mathbf{I}_{2m} + t\mathbf{J}_w) &= \text{Sdet}_t \begin{bmatrix} 1 & w(e_1)t & 0 & 0 & \cdots \\ w(e_1^{-1})t & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & w(e_2)t & 0 \\ 0 & 0 & w(e_2^{-1})t & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots \end{bmatrix} \\
&= \text{Sdet}_t \begin{bmatrix} 1 - w(e_1)w(e_1^{-1})t^2 & w(e_1)t & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 - w(e_2)w(e_2^{-1})t^2 & w(e_2)t & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \vdots & \vdots & \vdots & 0 & \ddots \end{bmatrix} \quad (7.18) \\
&= \prod_{i=1}^m (1 - w(e_i)w(e_i^{-1})t^2)(1 - w(e_i)w(e_i^{-1})t^2)^*.
\end{aligned}$$

Let $\mathbf{X}(e)$ be the 2×2 matrix defined as follows:

$$\mathbf{X}(e) = \begin{bmatrix} 1 & w(e)t \\ w(e^{-1})t & 1 \end{bmatrix}.$$

Then, the inverse of $\mathbf{X}(e)$ is given by

$$\mathbf{X}(e)^{-1} = \begin{bmatrix} (1 - w(e)w(e^{-1})t^2)^{-1} & -(1 - w(e)w(e^{-1})t^2)^{-1}w(e)t \\ -(1 - w(e^{-1})w(e)t^2)^{-1}w(e^{-1})t & (1 - w(e^{-1})w(e)t^2)^{-1} \end{bmatrix}.$$

Therefore,

$$(\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1} = \begin{bmatrix} \mathbf{X}(e_1)^{-1} & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{X}(e_2)^{-1} & & \vdots \\ \vdots & & \ddots & \mathbf{O} \\ \mathbf{O} & \cdots & \mathbf{O} & \mathbf{X}(e_m)^{-1} \end{bmatrix}.$$

If $e = (u, v) \in D(G)$, then we have by a calculation that:

$$\begin{aligned}
({}^T\mathbf{L}(\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1}\mathbf{K})_{uv} &= \sum_{f, f' \in D(G)} ({}^T\mathbf{L})_{uf} ((\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1})_{ff'} \mathbf{K}_{f'v} \\
&= ({}^T\mathbf{L})_{ue} ((\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1})_{ee} \mathbf{K}_{ev} \\
&= (1 - w(e)w(e^{-1})t^2)^{-1}w(e).
\end{aligned}$$

If $u = v$, then we have:

$$\begin{aligned}
({}^T\mathbf{L}(\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1}\mathbf{K})_{uu} &= \sum_{f, f' \in D(G)} ({}^T\mathbf{L})_{uf} ((\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1})_{ff'} \mathbf{K}_{f'u} \\
&= \sum_{\substack{e \in D(G) \\ o(e)=u}} ({}^T\mathbf{L})_{ue} ((\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1})_{ee^{-1}} \mathbf{K}_{e^{-1}u} \\
&= - \sum_{\substack{e \in D(G) \\ o(e)=u}} (1 - w(e)w(e^{-1})t^2)^{-1}w(e)w(e^{-1})t.
\end{aligned}$$

Hence, it follows that:

$${}^T\mathbf{L}(\mathbf{I}_{2m} + t\mathbf{J}_w)^{-1}\mathbf{K} = \tilde{\mathbf{W}} - t\tilde{\mathbf{D}}. \quad (7.19)$$

Combining (7.17), (7.18) and (7.19), we obtain:

$$\begin{aligned} & \text{Sdet}_t(\mathbf{I}_{2m} - t(\mathbf{B}_w - \mathbf{J}_w)) \\ &= \text{Sdet}_t(\mathbf{I}_n - t\tilde{\mathbf{W}} + t^2\tilde{\mathbf{D}}) \prod_{i=1}^m (1 - w(e_i)w(e_i^{-1})t^2)(1 - w(e_i)w(e_i^{-1})t^2)^*. \end{aligned}$$

Now, the assertion follows from Theorem 7.1. \square

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